RIZVI COLLEGE OF ARTS, SCIENCE AND COMMERCE
OFF CARTER ROAD, BANDRA WEST

FYBSC SEM II
PHYSICS PAPER I UNIT III
SUPERPOSITION OF COLLINEAR HARMONIC OSCILLATORS

PREPARED BY

PROF. VISHWAS DESHMUKH

ASSISTANT PROFESSOR

DEPTMARTMENT OF PHYSICS

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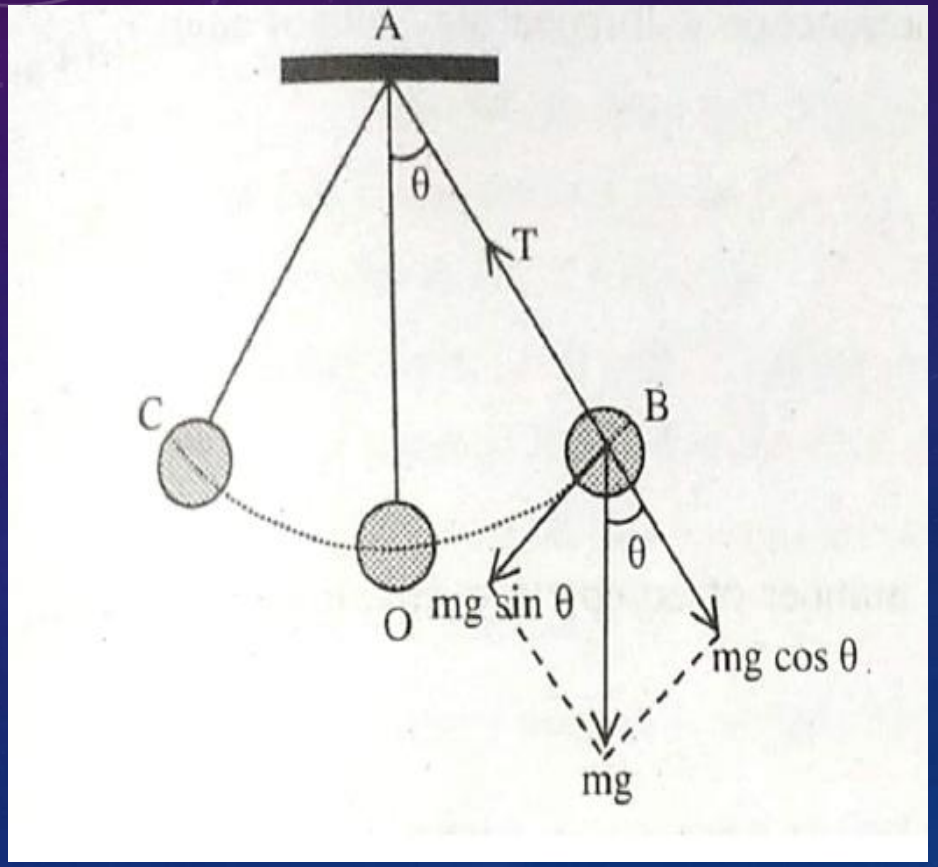
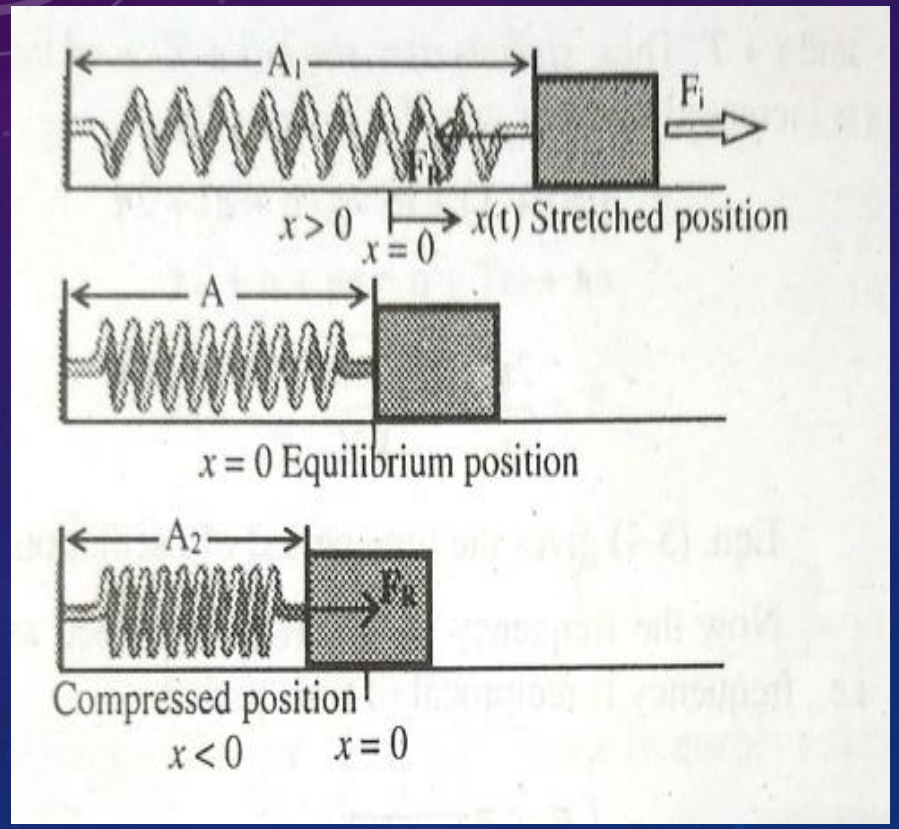
SIMPLE PENDULUM AS SIMPLE HARMONIC OSCILLATOR

1. Simple pendulum is a common example of simple harmonic motion.
2. A mass ‘m’ is suspended by an inextensible string and rest along the position OA as shown in the figure.
3. If pendulum is displaced through a small angle θ it begins to oscillates on two sides of its mean position.
4. In the displaced position ie at point B the forces acting on the pendulum bob are ; (I) its weight ‘mg’ vertically downwards and (ii) tension T in the string.
5. The weight can be resolved into its components. (a) mgcos θ along the string and (b) mgsinθ perpendicular to the string
6. The component mgcosθ is balanced by the tension in the string T
7. mgsinθ is unbalanced component of the force tends to restore the bob to its mean position.



SPRING MASS OSCILLATOR

1. Spring mass system on a frictionless surface is a SHO.
2. Spring has its spring constant K. and equilibrium length ‘A’
3. Select rest position as origin , marked as x= 0. when x > 0 spring is extended and when x < 0 spring is compressed.
4. The force acting on the spring is restoring force $F_R = - Kx$
5. Diagram shows the position of the mass when at rest , when spring is extended and when spring is compressed.



Second order differential equation for the linear simple harmonic motion, where the oscillator is oscillating along the positive x-direction

Let ‘m’ be the mass of the oscillator, executing SHM in X-direction.

Let X be the displacement for its mean position.

The velocity at instant t = $\frac{dx}{dt}$; and

Acceleration = $\frac{d^2x}{dt^2}$,

F_R be the restoring force acting in the direction opposite to the displacement.

Now, m $\frac{d^2x}{dt^2}$ is proportional to x;

Applying Newton's second law of motion ;

$$m\frac{d^2x}{dt^2} = -Kx \text{ where, K is the proportionality constant,}$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{K}{m}x$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2 x \qquad \left(\because \frac{K}{m} = \omega^2\right)$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

The above equation is second order differential equation of motion for SHO. The solution of the equation can be $\sin(\omega\, t + \alpha)$, $\cos(\omega\, t + \alpha)$ and /or linear combination

$$\boldsymbol{X = A\, sin(\omega\, t + \alpha)}$$

Where A is the max displacement from mean position. ω is angular frequency, α is called phase or epoch

Definitions

Amplitude : Quantity "A" is known as the amplitude of the motion. Amplitude is the maximum value of the displacement on either side from the equilibrium position. Since the maximum and minimum values are +1 and -1 the maximum ad minimum displacement are +A and –A.

Time period : The time interval during which the oscillations repeats itself is known as Time period.
The displacement of the particle performing SHM have same value ate time t and t + T.
Thus $\sin(\omega t + \alpha) = \sin[\omega(t + T) + \alpha]$
After one periodic time; $(\omega(t + T) + \alpha) = (\omega t + \alpha) + 2\pi$

$$\therefore \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}$$

Frequency : The number of complete oscillation per unit time.

$$\therefore \quad F = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

SUPERPOSITION OF TWO COLLINEAR SIMPLE HARMONIC MOTIONS OF SAME PERIOD AND HAVING SAME CENTRE.

(A) Superposition of Two Equal Frequencies

Let a particle is simultaneously subjected to two collinear Simple Harmonic Motions (SHMs) having same time period and same centre but of different amplitudes and phase constants (i.e different epoch angles).

Let the two SHMs represented by,

$$x_1 = A \sin(\omega t + \alpha) \quad \ldots(5.6)$$

$$x_2 = B \sin(\omega t + \beta) \quad \ldots(5.7)$$

where, x_1 and x_2 are displacements of particle along x-axis at any instant of time t, A and B are the amplitudes, α and β are phase constants and both SHMs have the same period ($T = \frac{2\pi}{\omega}$; ω being the angular frequency).

Since both the displacements of SHMs are along the same straight line, the resultant displacement (x) at that instant t is equal to the sum of two displacements (vector sum is here reduced to an algebraic sum).

By using the superposition principle, the resultant displacement x is given by,

$$\therefore \quad x = x_1 + x_2$$

$$= A \sin(\omega t + \alpha) + B \sin(\omega t + \beta)$$

By using trigonometry formula, $\sin(A + B) = \sin A \cos B + \cos A \sin B$,

$$\therefore \quad x = A(\sin \omega t \cos \alpha + \cos \omega t \sin \alpha) + B(\sin \omega t \cos \beta + \cos \omega t \sin \beta)$$

$$= A \sin \omega t \cos \alpha + A \cos \omega t \sin \alpha + B \sin \omega t \cos \beta + B \cos \omega t \sin \beta$$

$$= \sin \omega t (A \cos \alpha + B \cos \beta) + \cos \omega t (A \sin \alpha + B \sin \beta) \quad ...(5.8)$$

Let $R \cos \delta = A \cos \alpha + B \cos \beta$ ...(5.9)

$$R \sin \delta = A \sin \alpha + B \sin \beta \quad ...(5.10)$$

Substituting eqns. (5.9) and (5.10) into eqn. (5.8), we get,

$$x = \sin \omega t \, R \cos \delta + \cos \omega t \, R \sin \delta$$

$$\therefore \quad x = R (\sin \omega t \cos \delta + \cos \omega t \sin \delta)$$

$$\therefore \quad x = R \sin (\omega t + \delta) \quad ...(5.11)$$

Thus the resulting motion is simple harmonic with same period $T = \frac{2\pi}{\omega}$ (same frequency) along the same line and has the same centre as that of the constituent SHMs.

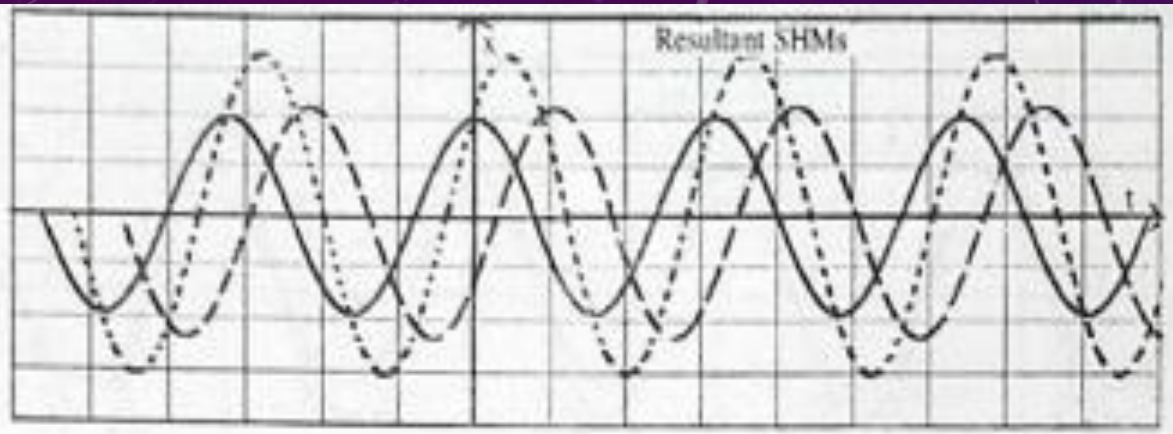


Fig. 5.1: Superposition of two collinear SHMs $A \neq B$ and $\delta = \frac{\pi}{2}$

The resultant amplitude R is obtained by squaring and adding the eqns. (5.9) and (5.10).

$$\therefore \quad (R\cos\delta)^2 + (R\sin\delta)^2 = (A\cos\alpha + B\cos\beta)^2 + (A\sin\alpha + B\sin\beta)^2 \qquad ...(5.12)$$

Solving L.H.S. of eqn. (5.12), we get,

$$\text{L.H.S.} = R^2\cos^2\delta + R^2\sin^2\delta = R^2(\sin^2\delta + \cos^2\delta) = R^2 \times 1 = R^2$$

Solving R.H.S. of eqn. (5.12), we get,

$$\text{R.H.S.} = (A^2\cos^2\alpha + 2AB\cos\alpha\cos\beta + B^2\cos^2\beta) + (A^2\sin^2\alpha + 2AB\sin\alpha\sin\beta + B^2\sin^2\beta)$$

$$= A^2\cos^2\alpha + 2AB\cos\alpha\cos\beta + B^2\cos^2\beta + A^2\sin^2\alpha + 2AB\sin\alpha\sin\beta + B^2\sin^2\beta$$

$$= A^2(\cos^2\alpha + \sin^2\alpha) + B^2(\cos^2\beta + \sin^2\beta) + 2AB\cos\alpha\cos\beta + 2AB\sin\alpha\sin\beta$$

$$= A^2 + B^2 + 2AB(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

By using trigonometry formula, $\cos(A - B) = \cos A \cos B + \sin A \sin B$,

$$\text{R.H.S.} = A^2 + B^2 + 2AB\cos(\alpha - \beta)$$

Substituting L.H.S. and R.H.S. in eqn. (5.12)

$$R^2 = A^2 + B^2 + 2AB\cos(\alpha - \beta) \quad \ldots(5.13)$$

where $\delta = \alpha - \beta$ = constant.

$$R^2 = A^2 + B^2 + 2AB\cos\delta \quad \ldots(5.14)$$

It is clear from the nature of eqns. (5.13) or (5.14) that the resultant amplitude R is the vector sum of A and B.

The resultant phase constant, i.e., the resultant epoch δ can be obtained by dividing eqn. (5.10) by eqn. (5.9).

$$\therefore \quad \tan\delta = \frac{R\sin\delta}{R\cos\delta} = \frac{A\sin\alpha + B\sin\beta}{A\cos\alpha + B\cos\beta} \quad \ldots(5.15)$$

SPECIAL CASES

Case 1: $\delta = \alpha - \beta = 2n\pi$, where $n = 0, 1, 2, \ldots$ Thus, δ is even integral multiple of π. Note that $\cos n\pi = (-1)^n$, $\cos(\alpha - \beta) = 1$.

From eqn. (5.14),

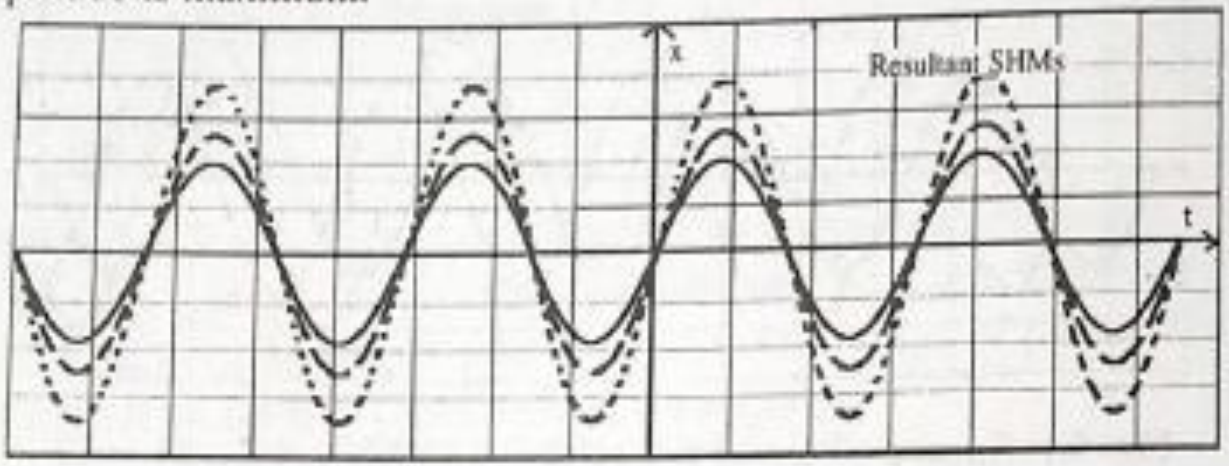
$$R^2 = A^2 + B^2 + 2AB \cos\delta$$

$$\therefore \quad R^2 = A^2 + B^2 + 2AB$$

$$\therefore \quad R^2 = (A + B)^2$$

$$\therefore \quad R = A + B \qquad \ldots(5.16)$$

When the two superposed SHMs are in the same phase and resultant SHM has also same phase. The resultant amplitude is maximum.



Case 2: $\delta = \alpha - \beta = (2n + 1)\pi$, where $n = 0, 1, 2,...$ Thus, δ is odd integral multiple of π. Note that $\cos n\pi = (-1)^n$, $\cos(\alpha - \beta) = -1$.

From eqn. (5.14),

$$R^2 = A^2 + B^2 + 2AB\cos\delta$$

$$\therefore \quad R^2 = A^2 + B^2 - 2AB$$

$$\therefore \quad R^2 = (A - B)^2$$

$$\therefore \quad R = A - B \qquad ...(5.17)$$

When the two SHMs are in the opposite phase. The resultant amplitude is minimum.

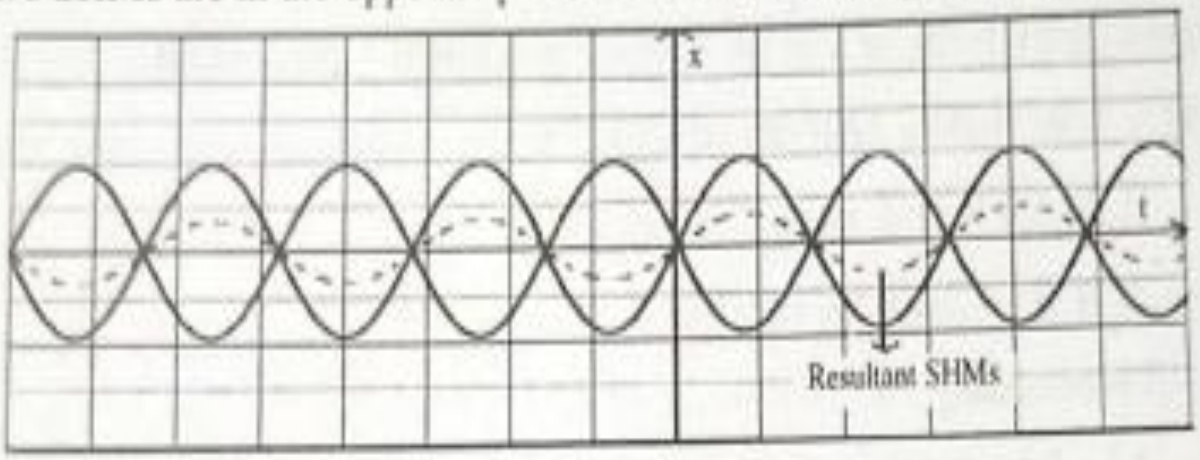


Fig. 5.3: Superposition of two collinear SHMs in opposite phase

Case 3. (a) If A = B and α and β are different then

$$R^2 = A^2 + A^2 + 2AA\cos(\alpha - \beta)$$
$$= 2A^2[1 + \cos(\alpha - \beta)]$$
$$= 2A^2\left[2\cos^2\left(\frac{\alpha-\beta}{2}\right)\right]$$
$$\therefore \quad R = 2A\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\text{Also; } \tan\delta = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$
$$= \tan\left(\frac{\alpha+\beta}{2}\right)$$
$$\therefore \quad \delta = \left(\frac{\alpha+\beta}{2}\right)$$

Case 3 (b) : If A= B and δ = α - β = 2nπ , where n = 0,1,2….

Then R = R_{max} = A + A = 2A

Case 3 (c) : If A= B and δ = α - β = (2n+ 1)π where n = 0,1,2,3…

Then R = R_{min} = 0

Superposition of two different frequencies (beats)

Beats can be defined as the periodic variation of intensity of the wave at a given point due to superposition of two waves of slightly different frequencies travelling along the same direction.

Two simple harmonic oscillators are ;

$$y_1 = a_1 \sin(\omega t + \alpha_1)$$

$$y_2 = a_2 \sin\{(\omega + \Delta\omega)t + \alpha_2\}$$

$$\therefore \quad y_2 = a_2 \sin(\omega t + \Delta\omega t + \alpha_2)$$

Let $(\Delta\omega t + \alpha_2) = \alpha_2'$

$$\therefore \quad y_2 = a_2 \sin(\omega t + \alpha_2') \qquad \ldots(5.22)$$

Thus, resultant displacement at any instant is, $y = y_1 + y_2$

$$y = a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2') \qquad \ldots(5.23)$$

By using trigonometric formula, $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$y = a_1(\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + a_2(\sin \omega t \cos \alpha_2' + \cos \omega t \sin \alpha_2')$$

$$y = a_1 \sin \omega t \cos \alpha_1 + a_1 \cos \omega t \sin \alpha_1 + a_2 \sin \omega t \cos \alpha_2' + a_2 \cos \omega t \sin \alpha_2'$$

Rearranging the terms,

$$y = \sin \omega t (a_1 \cos \alpha_1 + a_2 \cos \alpha_2') + \cos \omega t (a_1 \sin \alpha_1 + a_2 \sin \alpha_2') \qquad \ldots(5.24)$$

Let,

$$A \cos \delta = a_1 \cos \alpha_1 + a_2 \cos \alpha_2' \qquad \ldots(5.25)$$

$$A \sin \delta = a_1 \sin \alpha_1 + a_2 \sin \alpha_2' \qquad \ldots(5.26)$$

By using eqn. (5.24), (5.25) and (5.26), we get,

$$\therefore \quad y = A \sin(\omega t + \delta) \qquad \ldots(5.27)$$

$$\tan \delta = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2'}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2'}$$

$$\therefore \quad \delta = \tan^{-1}\left(\frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2'}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2'}\right) \qquad \ldots(5.28)$$

The resultant motion given by eqn. (5.27) is not S.H.M. because the values of 'a' and 'δ' are not constant but they depend on time.

When, $\alpha_1 - \alpha_2' = 2n\pi$, $\quad n = 0, 1, 2, 3...$

$\cos(\alpha_1 - \alpha_2') = 1$

$$\therefore \quad A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \quad \text{or} \quad A = a_1 + a_2 \qquad ...(5.29)$$

Also when, $\alpha_1 - \alpha_2' = (2n + 1)\pi$, $\quad n = 0, 1, 2, 3...$

$\cos(\alpha_1 - \alpha_2') = -1$

$$\therefore \quad A^2 = a_1^2 + a_2^2 - 2a_1 a_2 \quad \text{or} \quad A = a_1 - a_2 \qquad ...(5.30)$$

$\therefore$ The amplitude of the resultant changes between $a_1 + a_2$ and $a_1 - a_2$. If $a_1 = a_2 = a$ then $A = 2a$ or 0. Thus the amplitude of the resultant vibration changes with time $\frac{2\pi}{\Delta\omega}$, i.e., with frequencies $\frac{\Delta\omega}{2\pi}$ which is the difference between the two component frequencies ω_1 and ω_2.

When the case is applied to sound, we get waxing and waning. This phenomenon is known as Beats.

SUPERPOSITION OF TWO MUTUALLY PERPENDICULAR HARMONIC MOTIONS OF SAME PERIOD AND SAME CENTRE

(A) Equal Frequency

Let a particle be subjected simultaneously to two mutually perpendicular SHMs of equal time periods but of different amplitudes and different phase constants and one of them acting along the x-axis and other acting along the y-axis. The two SHMs having the same centre, say the origin.

The particle moves in a plane in such a way that its two co-ordinates x and y oscillates with SHMs and its equations represents as follows:

$$x = A \sin(\omega t + \alpha) \quad \ldots(5.31)$$

and

$$y = B \sin(\omega t + \beta) \quad \ldots(5.32)$$

where, ω is the angular frequency, A and B are amplitudes, α and β are the phase constants and x and y are the displacements of two SHMs.

$$\therefore \quad \frac{x}{A} = \sin(\omega t + \alpha) = \sin \omega t \cos \alpha + \cos \omega t \sin \alpha \quad \ldots(5.33)$$

$$\therefore \quad \frac{y}{B} = \sin(\omega t + \beta) = \sin \omega t \cos \beta + \cos \omega t \sin \beta \quad \ldots(5.34)$$

Multiplying eqn. (5.33) by sin β, and we get,

$$\frac{x}{A} \sin \beta = \sin \omega t \cos \alpha \sin \beta + \cos \omega t \sin \alpha \sin \beta \quad \ldots(5.35)$$

also multiplying eqn. (5.34) by sin α, and we get,

$$\frac{y}{B} \sin \alpha = \sin \omega t \cos \beta \sin \alpha + \cos \omega t \sin \beta \sin \alpha \quad \ldots(5.36)$$

Subtracting eqn. (5.36) from eqn. (5.35) we get,

$$\left(\frac{x}{A}\sin\beta - \frac{y}{B}\sin\alpha\right)$$

$$= (\sin\omega t\cos\alpha\sin\beta + \cos\omega t\sin\alpha\sin\beta) - (\sin\omega t\cos\beta\sin\alpha + \cos\omega t\sin\beta\sin\alpha)$$

$$\therefore \quad \left(\frac{x}{A}\sin\beta - \frac{y}{B}\sin\alpha\right)$$

$$= \sin\omega t(\cos\alpha\sin\beta - \cos\beta\sin\alpha) + \cos\omega t(\sin\alpha\sin\beta - \sin\beta\sin\alpha)$$

$$\therefore \quad \left(\frac{x}{A}\sin\beta - \frac{y}{B}\sin\alpha\right) = \sin\omega t(\cos\alpha\sin\beta - \cos\beta\sin\alpha)$$

By using trigonometric formula $\sin(A - B) = \sin A\cos B - \cos A\sin B$, we get,

$$\therefore \quad \left(\frac{x}{A}\sin\beta - \frac{y}{B}\sin\alpha\right) = \sin\omega t\sin(\beta - \alpha) \qquad \ldots(5.37)$$

Squaring the eqn. (5.37)

$$\therefore \quad \left(\frac{x}{A}\sin\beta - \frac{y}{B}\sin\alpha\right)^2 = \sin^2\omega t \sin^2(\beta-\alpha)$$

$$\frac{x^2}{A^2}\sin^2\beta + \frac{y^2}{B^2}\sin^2\alpha - \frac{2xy}{AB}\sin\beta\cdot\sin\alpha = \sin^2\omega t\cdot\sin^2(\beta-\alpha) \qquad ...(5.38)$$

Similarly, multiplying eqn. (5.33) by cos β and eqn. (5.34) by cos α

$$\therefore \quad \frac{x}{A}\cos\beta = \sin\omega t\cos\alpha\cos\beta + \cos\omega t\sin\alpha\cos\beta \qquad ...(5.39)$$

$$\therefore \quad \frac{y}{B}\cos\alpha = \sin\omega t\cos\beta\cos\alpha + \cos\omega t\sin\beta\cos\alpha \qquad ...(5.40)$$

Subtracting eqn. (5.40) from eqn. (5.39) we get,

$$\therefore \quad \left(\frac{x}{A}\cos\beta - \frac{y}{B}\cos\alpha\right) = \cos\omega t\,(\sin\alpha\cos\beta - \sin\beta\cos\alpha)$$

$$\therefore \quad \left(\frac{x}{A}\cos\beta - \frac{y}{B}\cos\alpha\right) = -\cos\omega t\sin(\beta-\alpha) \qquad ...(5.41)$$

Squaring eqn. (5.41)

$$\therefore \quad \left(\frac{x}{A}\cos\beta - \frac{y}{B}\cos\alpha\right)^2 = \cos^2\omega t \sin^2(\beta-\alpha)$$

$$\frac{x^2}{A^2}\cos^2\beta + \frac{y^2}{B^2}\cos^2\alpha - \frac{2xy}{AB}\cos\beta\cdot\cos\alpha = \cos^2\omega t\cdot\sin^2(\beta-\alpha) \qquad ...(5.42)$$

Adding eqn.(5.38) and eqn. (5.42), we get,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}(\sin\beta\sin\alpha + \cos\beta\cos\alpha) = \sin^2(\beta-\alpha)(\sin^2\omega t + \cos^2\omega t)$$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos(\beta-\alpha) = \sin^2(\beta-\alpha) \qquad ...(5.43)$$

Let $\delta = (\beta-\alpha)$ = phase difference between the two component SHMs.

$$\therefore \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\delta = \sin^2\delta \qquad ...(5.44)$$

This is the equation of resultant path followed by the particle. The path will have different shapes depending on phase difference. Above equation is a general equation of an ellipse inclined to co-ordinates axes and it is described in a rectangle whose sides are given by 2A ($x = \pm A$) and 2B ($y = \pm B$). The centre of ellipse coincides with the origin.

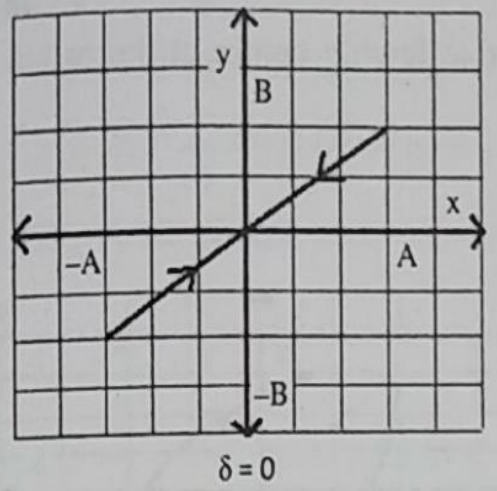


Fig. 5.6

Case 1. Let $\delta = \beta - \alpha = 0$

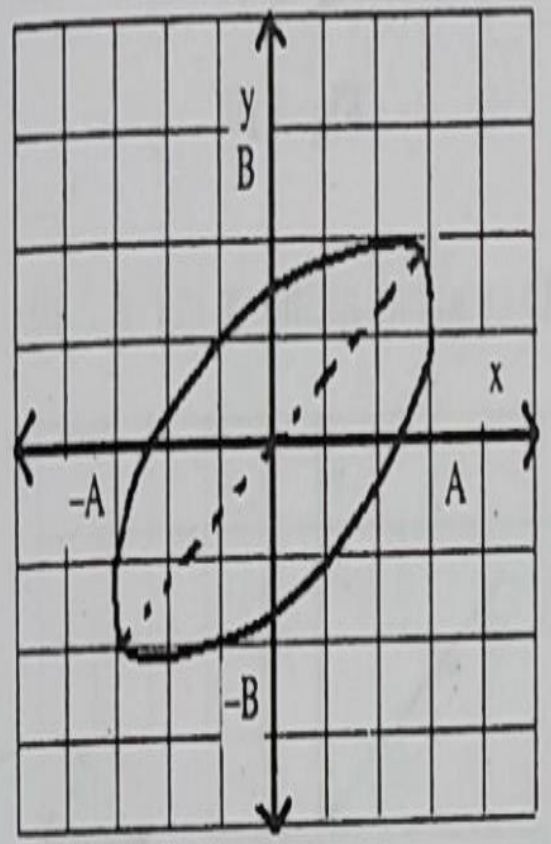
$\therefore \quad \cos\delta = 1$ and $\sin\delta = 0$.

$$\therefore \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB} = 0$$

$$\left(\frac{x}{A} - \frac{y}{B}\right)^2 = 0$$

$$\therefore \pm\left(\frac{x}{A} - \frac{y}{B}\right) = 0 \qquad \ldots(5.45)$$

$\frac{x}{A} = \frac{y}{B}$ for the positive sign in eqn. (5.45) gives a straight line in the first quadrant. $-\frac{x}{A} = -\frac{y}{B}$ for the negative sign in eqn. (5.45) gives straight line in the third quadrant



$\delta = \frac{\pi}{4}$

Fig. 5.7

Case 2. Let $\delta = \beta - \alpha = \frac{\pi}{4}$

$$\therefore \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

From eqn. (5.44), we get,

$$\therefore \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{\sqrt{2}xy}{AB} = \frac{1}{2} \qquad \ldots(5.46)$$

This represents an oblique ellipse shown in Fig. (5.7). It lies in the first and third quadrant.

Case 3. Let $\delta = \beta - \alpha = \frac{\pi}{2}$

$\therefore \cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1.$

From eqn.(5.44), we get,

$$\therefore \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \qquad \ldots(5.47)$$

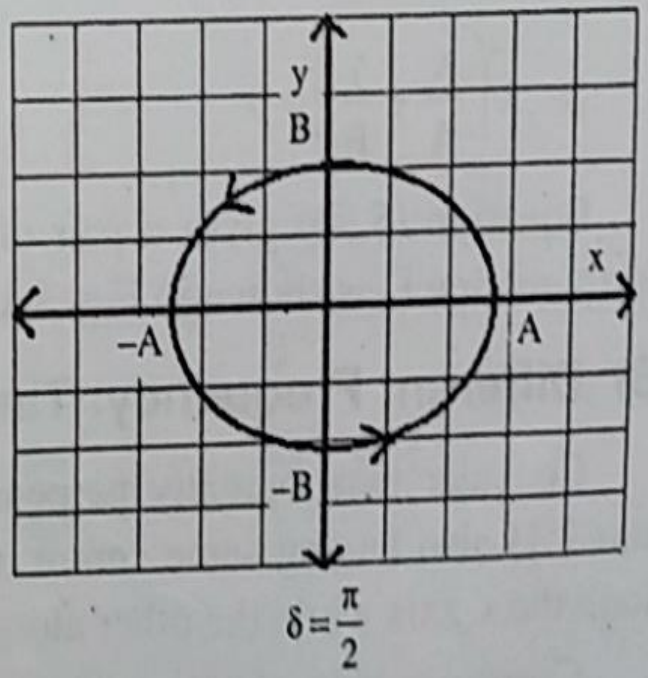
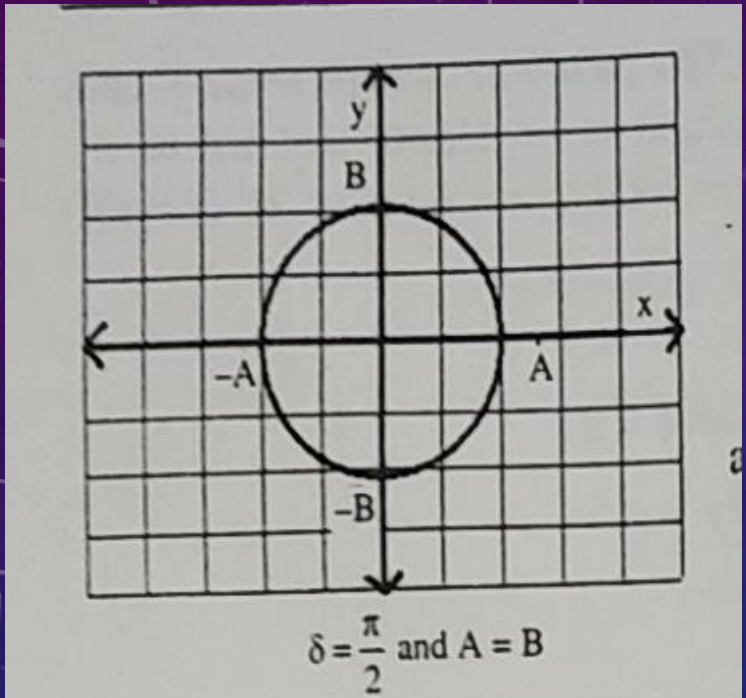


Fig. 5.8

This is the equation of a symmetrical ellipse, A and B as the semi major and semi minor axes respectively. The major and minor axis coincides with the coordinates X and Y.



Case 4 : let $\delta = \alpha - \beta = \frac{\pi}{2}$ and A = B

then we have

$$X^2 + Y^2 = A^2$$

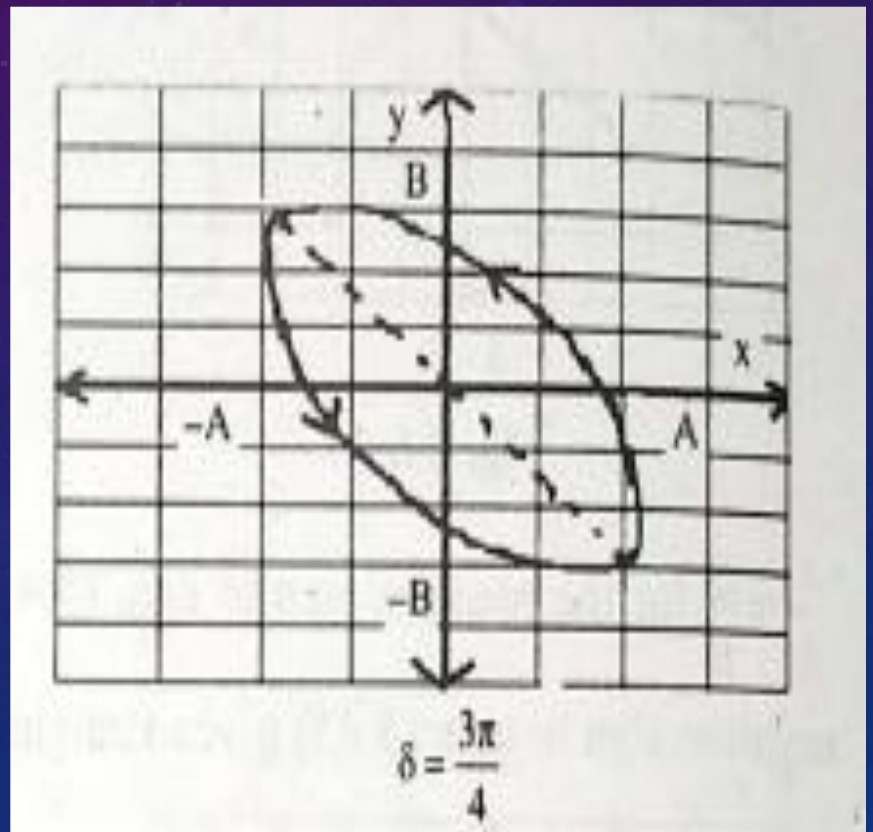
This is the equation of the circle having centre at the origin O and has radius A.

Case 5. Let $\delta = \beta - \alpha = \frac{3\pi}{4}$ or 135°

$$\therefore \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} \text{ and } \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

From eqn. (5.44) we get,

$$\therefore \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{\sqrt{2}xy}{AB} = \frac{1}{2} \qquad \ldots(5.49)$$



Case 6. Let $\delta = \beta - \alpha = \pi$

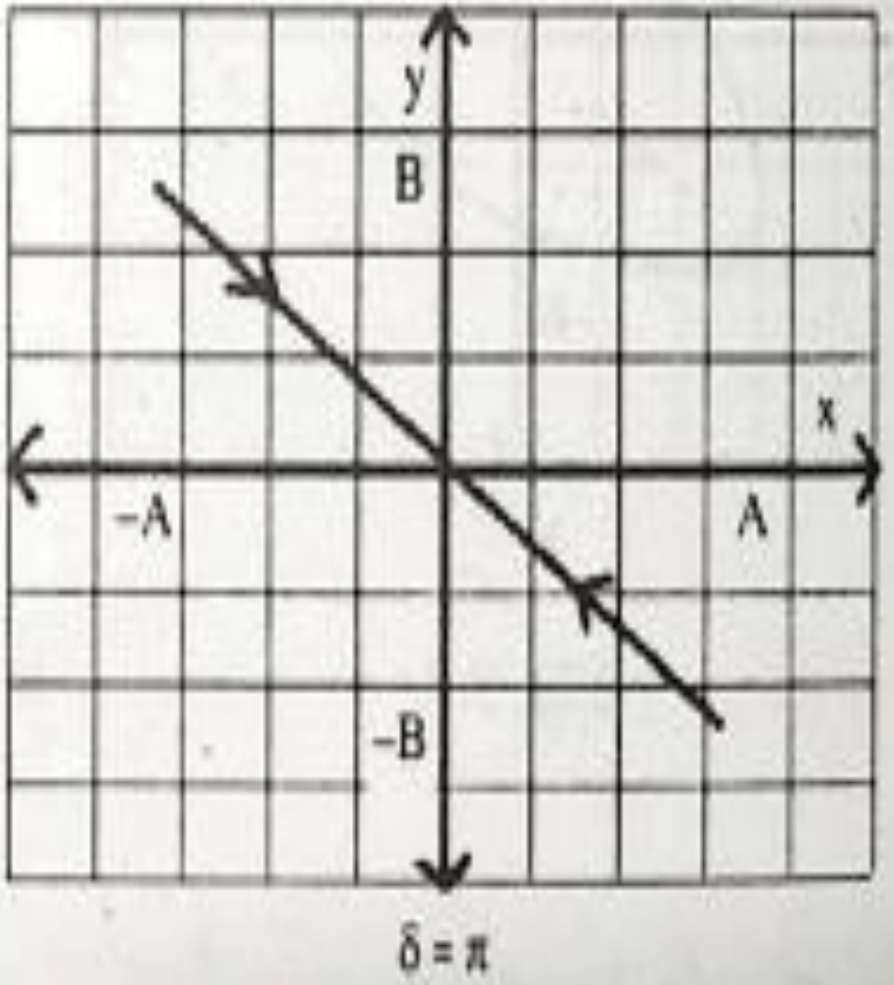
$\therefore \cos \pi = -1$ and $\sin \pi = 0$.

From eqn. (5.44)

$$\therefore \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{2xy}{AB} = 0$$

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 = 0$$

$$\therefore \pm\left(\frac{x}{A} + \frac{y}{B}\right) = 0$$



$\delta = \pi$

MUTUALLY PERPENDICULAR OSCILLATORS: DIFFERENT FREQUENCIES (TIME PERIOD IN THE RATIO 1:2)

$$x = A \sin(2\omega t + \alpha) \quad \text{...(5.51)}$$

$$y = B \sin(\omega t + \beta) \quad \text{...(5.52)}$$

where, A and B are amplitudes, α and β are phase constants, x and y are the displacements of the particle due two SHMs measured from the common origin.

Rearranging eqn. (5.51) and using $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$, we get,

$$\frac{x}{A} = \sin(2\omega t + \alpha) = \sin 2\omega t \cos\alpha + \cos 2\omega t \sin\alpha$$

$$\therefore \quad \frac{x}{A} = (2\sin\omega t \cos\omega t)\cos\alpha + (1 - 2\sin^2\omega t)\sin\alpha \quad \text{...(5.53)}$$

Similarly from eqn. (5.52), we get,

$$\therefore \quad \frac{y}{B} = \sin(\omega t + \beta) = \sin \omega t \cos \beta + \cos \omega t \sin \beta \qquad \ldots(5.54)$$

The phase difference $\delta = \alpha - \beta$, for solving the equation we assume that $\beta = 0$, then $\delta = \alpha$.

$$\therefore \quad \frac{x}{A} = 2\sin \omega t (1 - \sin^2 \omega t)^{\frac{1}{2}} \cos \delta + (1 - 2\sin^2 \omega t) \sin \delta \qquad (5.55)$$

and $\frac{y}{B} = \sin \omega t$ and substitute this in eqn. (5.55), to eliminate time t

$$\therefore \quad \frac{x}{A} = 2\left(\frac{y}{B}\right)\left(1 - \frac{y^2}{B^2}\right)^{\frac{1}{2}} \cos \delta + \left(1 - 2\frac{y^2}{B^2}\right) \sin \delta$$

$$\frac{x}{A} = 2\left(\frac{y}{B}\right)\left(1 - \frac{y^2}{B^2}\right)^{\frac{1}{2}} \cos \delta + \sin \delta - 2\frac{y^2}{B^2} \sin \delta$$

$$\therefore \quad \frac{x}{A} - \sin \delta + \frac{2y^2}{B^2} \sin \delta = 2\left(\frac{y}{B}\right)\left(1 - \frac{y^2}{B^2}\right)^{1/2} \cos \delta$$

$$\left[\left(\frac{x}{A} - \sin \delta\right) + \frac{2y^2}{B^2} \sin \delta\right] = \frac{2y}{B}\left(1 - \frac{y^2}{B^2}\right)^{\frac{1}{2}} \cos \delta$$

Squaring on both sides, we get,

$$\therefore \quad \left[\left(\frac{x}{A} - \sin \delta\right) + \frac{2y^2}{B^2} \sin \delta\right]^2 = \frac{4y^2}{B^2}\left(1 - \frac{y^2}{B^2}\right) \cos^2 \delta$$

Rearranging the RHS side,

$$\therefore \quad \left[\left(\frac{x}{A}-\sin\delta\right)+\frac{2y^2}{B^2}\sin\delta\right]^2=\frac{4y^2}{B^2}\cos^2\delta-\frac{4y^4}{B^4}\cos^2\delta$$

$$\therefore \quad \left(\frac{x}{A}-\sin\delta\right)^2+2\left(\frac{x}{A}-\sin\delta\right)\frac{2y^2}{B^2}\sin\delta+\frac{4y^4}{B^4}\sin^2\delta=\frac{4y^2}{B^2}\cos^2\delta-\frac{4y^4}{B^4}\cos^2\delta$$

$$\therefore \quad \left(\frac{x}{A}-\sin\delta\right)^2+\frac{4xy^2}{AB^2}\sin\delta-\frac{4y^2}{B^2}\sin^2\delta+\frac{4y^4}{B^4}\sin^2\delta-\frac{4y^2}{B^2}\cos^2\delta+\frac{4y^4}{B^4}\cos^2\delta=0$$

$$\therefore \quad \left(\frac{x}{A}-\sin\delta\right)^2+\frac{4y^4}{B^4}\left(\sin^2\delta+\cos^2\delta\right)+\frac{4xy^2}{AB^2}\sin\delta-\frac{4y^2}{B^2}\left(\sin^2\delta+\cos^2\delta\right)=0 \qquad ...(5.56)$$

By using trigonometry formula $\sin^2\delta+\cos^2\delta=1$, substitute into the eqn.(5.56), we get,

$$\therefore \quad \left(\frac{x}{A}-\sin\delta\right)^2+\frac{4y^4}{B^4}+\frac{4xy^2}{AB^2}\sin\delta-\frac{4y^2}{B^2}=0$$

$$\therefore \left(\frac{X}{Y} - \sin\delta\right)^2 + \frac{4Y^2}{B^2}\left(\frac{Y^2}{B^2} + \frac{X}{A}\sin\delta - 1\right) = 0$$

Putting the value $\delta = \alpha - \beta$ in the above equation;

$$\left(\frac{X}{Y} - \sin(\alpha - \beta)\right)^2 + \frac{4Y^2}{B^2}\left(\frac{Y^2}{B^2} + \frac{X}{A}\sin(\alpha - \beta) - 1\right) = 0$$

This represents the general equation of a curve with tow loops

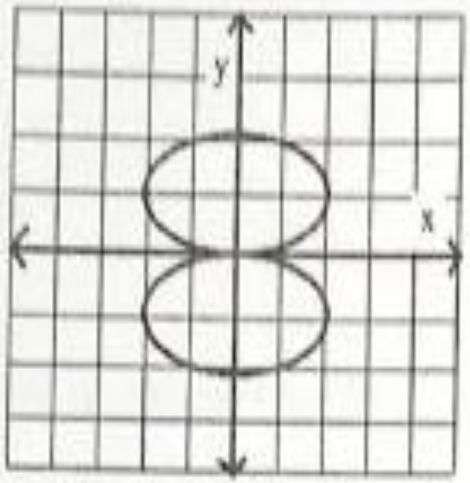
the resultant motion depends on the phase difference δ

Case 1: When $\delta = \alpha - \beta = 0, \pi, 2\pi \ldots \ldots; \quad \sin\delta = 0$

The eqn. (5.57) becomes,

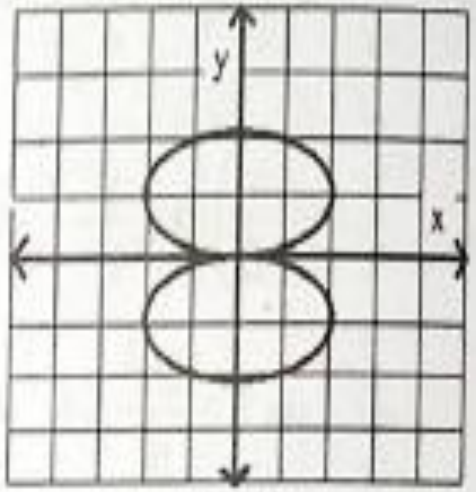
$$\therefore \left(\frac{x^2}{A^2}\right) + \frac{4y^2}{B^2}\left(\frac{y^2}{B^2} - 1\right) = 0 \qquad \ldots(5.58)$$

The above equation represents a figure of eight, i.e., shown in Fig. 5.12(a) and 5.12(b).



$\delta = 0$

(a)



$\delta = \pi$

(b)

Case 2: When $\delta = \alpha - \beta = \frac{\pi}{4}$; $\sin\delta = \frac{\pi}{4}$

The eqn. (5.57) becomes,

$$\left(\frac{x}{A} - \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{B^2}\left(\frac{y^2}{B^2} + \frac{x}{\sqrt{2}A} - 1\right) = 0$$

Also, When $\delta = \alpha - \beta = \frac{3\pi}{4}$;

[**Note:** $\sin\delta = \sin\left(\frac{3\pi}{4}\right) = \sin(135^\circ)$

$\sin(135^\circ) = \sin(90^\circ + 45^\circ)$, By using trignometric

$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \cos B$

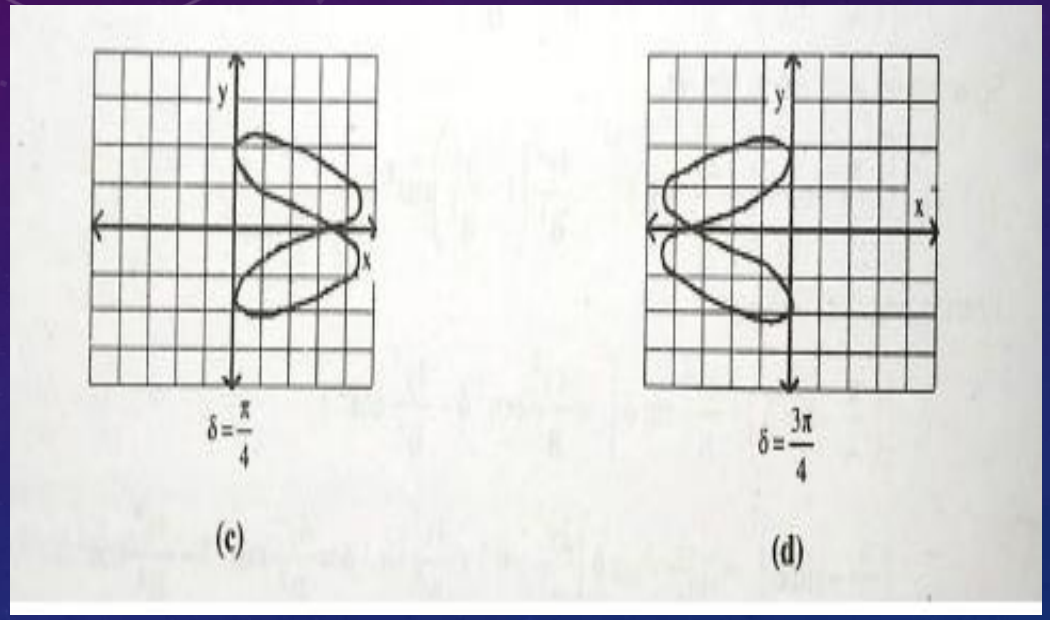
$\sin(90^\circ + 45^\circ) = \sin 90^\circ \cos 45^\circ + \cos 90^\circ \sin 45^\circ$

$$\sin(135^\circ) = \left(1 \times \frac{1}{\sqrt{2}}\right) + \left(0 \times 1/\sqrt{2}\right)$$

$\sin(135^\circ) = \frac{1}{\sqrt{2}}$ or $\sin(135^\circ) = \sin(90^\circ + 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$]

The eqn. (5.57) becomes,

$$\left(\frac{x}{A} - \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{B^2}\left(\frac{y^2}{B^2} + \frac{x}{\sqrt{2}A} - 1\right) = 0$$



Case 3: When $\delta = \alpha - \beta = \frac{\pi}{2}$; $\sin\delta = 1$

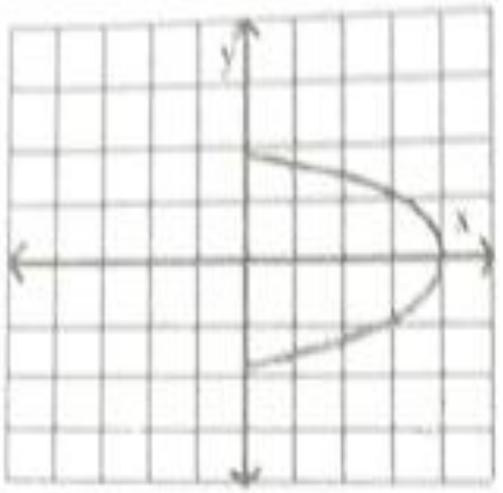
The eqn. (5.57) becomes,

$$\left(\frac{x}{A}-1\right)^2+\frac{4y^2}{B^2}\left(\frac{y^2}{B^2}+\frac{x}{A}-1\right)=0$$

$$\left[\left(\frac{x}{A}-1\right)+\left(\frac{2y^2}{B^2}\right)\right]^2=0$$

This equation represents two coincidents parabolas and each of which has the equation,

$$\left(\frac{x}{A}-1\right)^2+\frac{2y^2}{B^2}=0 \quad \text{OR} \quad y^2=-\frac{B^2}{2A}(x-A)$$



THANK YOU